Off-policy Monte Carlo agents with variable behaviour policies

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Abstract. This paper looks at the convergence property of off-policy Monte Carlo agents with variable behaviour policies. It presents results about convergence and lack of convergence. Even if the agent generates every possible episode history infinitely often, the algorithm can fail to converge on the correct Q-values. On the other hand, it can converge on the correct Q-values under certain conditions. For instance, if, during the n-th episode, the agent has an independent probability of $1/\log(n)$ of following the original policy at any given state, then it will converge on the right Q-values for that policy.

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1 Off-policy Monte Carlo

The Monte Carlo agent is a model-free reinforcement learning agent [3]. These operate when the environment is a Markov decision process (MDP). In an MDP, the next observation depends only on the current observation – the state – and the current action. The full set of state action pairs is designated by $\mathcal{S} \times \mathcal{A}$.

In each state, an agent chooses its actions according to a (possibly stochastic) policy π, that is assumed to be Markov. Monte Carlo agents operate by computing the Q-values of a policy for each state-action pair (s, a), which is the expected return if the agent choose a in s and subsequently follows π. It is episodic, exploring the same MDP repeatedly to compute Q-values. Episodic MDP's have initial state (where the agent starts an episode) and terminal states (where the agent ends an episode). This paper focuses on computing Q-values, not on updating policy choice in consequence.

Assumption 1 *In the following, these are always assumed, required for convergence results:*

(a) The MDP is finite,
(b) Whatever policy the agent uses, its expected time until reaching a terminal state is finite.
(c) The rewards after each step have finite expectation (consequently the total reward has finite expectation).

If an agent is following one policy (*e.g.*, ρ) but wishes to compute the Q-values of another (*e.g.*, π), it can use the off-policy Monte Carlo algorithm [3]. In this case ρ is the *behaviour policy*, and π is the *evaluation policy*.

The algorithm requires that for all state-action pairs (s, a):

$$\pi(a|s) > 0 \Rightarrow \rho(a|s) > 0. \tag{1}$$

Because of this requirement, $\pi(a|s)$ is zero whenever $\rho(a|s)$ is, and the ratio $\pi(a|s)/\rho(a|s)$ can be defined as 0 in these cases. The full algorithm is then given in box 1 – there are two variants, *ordinary importance sampling* using $N(s, a)$ (the number of episodes in which the agent has encountered the state-action pair (s, a)) as the denominator, and *weighted importance sampling* using $W(s, a)$ (the sum of the weights) instead [4, 1].

1. For all $s \in \mathcal{S}$ and $a \in \mathcal{A}_s$, initialise $N(s, a)$, $W(s, a)$ and $WR(s, a)$ to zero.
2. For all $n \geq 1$:
   (a) Generate an episode history h_n by following policy ρ.
   (b) The episode data will consist of the state, action chosen in the state, the immediate reward experienced by the agent.
   (c) For each state-action pair (s, a) appearing in the episode history:
      i. Let t be the first appearance of (s, a) in the history.
      ii. Let $R_n(s, a)$ be the total subsequent reward.
      iii. Define
      $$w_n(s, a) = \prod_{k>t} \frac{\pi(a_k|s_k)}{\rho(a_k|s_k)}. \tag{2}$$
      iv. $N_n(s, a) = N_{n-1}(s, a) + 1$.
      v. $W_n(s, a) = W_{n-1}(s, a) + w_n(s, a)$.
      vi. $WR_n(s, a) = WR_{n-1}(s, a) + w_n(s, a)R(s, a)$.
      vii. *Either* $Q_n(s, a) = \frac{WR_n(s,a)}{N_n(s,a)}$, *or* $Q_n(s, a) = \frac{WR_n(s,a)}{W_n(s,a)}$.
   (d) Discard the episode data.

Table 1. Off-policy Monte Carlo algorithm

Note that $W(s, a)$ and $N(s, a)$ have the same expectation: let $\mathcal{H}_{(s,a)}$ be the set of possible histories in the MDP subsequent to (s, a). For $h \in \mathcal{H}_{(s,a)}$, relabel the indexes so that h starts in step zero at (s,a). Use ρ and π to denote the probabilities of certain events conditional on the agent following those policies, and E_ρ and E_π similarly. Then if $I_n(s,a)$ denotes the event that (s,a) appears in the n-the episode,

$$\begin{aligned}
\mathrm{E}_\rho(w(s,a)|I_n(s,a)) &= \sum_{h\in\mathcal{H}_{(s,a)}} \rho(h|I_n(s,a)) \prod_{k>0} \frac{\pi(h_{a_k}|h_{s_k})}{\rho(h_{a_k}|h_{s_k})} \\
&= \sum_{h\in\mathcal{H}_{(s,a)}} \rho(h|I_n(s,a)) \frac{\pi(h|I_n(s,a))}{\rho(h|I_n(s,a))} \\
&= \sum_{h\in\mathcal{H}_{(s,a)}} \pi(h|I_n(s,a)) \\
&= \pi(\mathcal{H}_{(s,a)}|I_n(s,a)) = 1.
\end{aligned} \tag{3}$$

where h_{a_k} denotes the k-th action of history h, and h_{s_k} denotes the k-th state of history h.

Then simply note that the value of $N_n(s,a)$ is simply the sum $\sum_{j=1}^n I^j_{(s,a)}$, to see that $N_n(s,a)$ and $W_n(s,a) = \sum_{j=1}^n I^j_{(s,a)} w_n(s,a)$ have same expectation. If ρ is fixed, then by sampling only those histories including (s,a), the strong law of large numbers implies that the ratio of $N_n(s,a)$ and $W_n(s,a)$ converges to 1 almost surely.

The same argument as in equation (3) shows that

$$\begin{aligned}
\mathrm{E}_\rho(w_n(s,a)R_n(s,a)|I_n(s,a)) &= \sum_{h\in\mathcal{H}_{(s,a)}} R^h(s,a)\rho(h|I_n(s,a)) \prod_{k>0} \frac{\pi(h_{a_k}|h_{s_k})}{\rho(h_{a_k}|h_{s_k})} \\
&= \mathrm{E}_\pi(R(s,a)|I_n(s,a)),
\end{aligned} \tag{4}$$

where $R^h(s,a)$ is the reward along the history h subsequent to the first (s,a). Thus the expected *weighted* reward from following policy ρ, is the expected reward from following policy π.

Assume that agents following ρ would almost surely explore every state-action pair infinitely often (equivalently, that $N_n(s,a) \to \infty$ almost surely). Then the convergence of ordinary importance sampling is simply a consequence of the law of large numbers applied to every episode that visits (s,a). Convergence of weighted importance sampling is a consequence of Corollary 1 below.

2 Varying the behaviour policy

The previous proof, however, assumes that policy ρ is fixed. But what if it varies from episode to episode?

If π and π' are two policies, any $p \in [0,1]$ defines the mixed policy:

$$(1-p)\pi + p\pi'.$$

This is the policy that, in each state, independently chooses whether to follow π, with probability $1-p$, and π', with probability p. Because the decisions are independent, the mixed policy is Markov if π and π' are.

A general form for all behaviour policies is then:

Proposition 1. *If ρ and π obey the restriction in equation (1), then there exists a $\theta \in [0,1)$ and a policy π' such that*

$$\rho = (1-\theta)\pi + \theta\pi'.$$

Proof. Let σ be the maximum across $\mathcal{S} \times \mathcal{A}$ of the ratio $\pi(a|s)/\rho(a|s)$, in all cases where $\rho(a|s) \neq 0$. Then define $(1-\theta) = 1/\sigma$. Since $\pi(a|s)/\rho(a|s) \leq \sigma$, $(1-\theta) = 1/\sigma \leq \rho(a|s)/\pi(a|s)$ and hence $\rho(a|s) \geq (1-\theta)\pi(a|s)$.

Then define π' as

$$\pi'(a|s) = \frac{1}{\theta}\Big(\rho(a|s) - (1-\theta)\pi(a|s)\Big).$$

To check that this quantity is less than 1, note that $\rho(a|s) = 1 - \sum_{b\neq a}\rho(b|s) \leq 1 - \sum_{b\neq a}(1-\theta)\pi(b|s)$ hence that $\rho(a|s) - (1-\theta)\pi(a|s) \leq 1 - \sum_{b}(1-\theta)\pi(b|s) \leq 1-(1-\theta) \leq \theta$.

Then, by construction, $\rho(a|s) = (1-\theta)\pi(a|s) + (\theta)\pi'(a|s)$. □

For n being the episode number, define ρ_n by

$$\rho_n = (1-\theta_n)\pi + \theta_n\pi'_n, \tag{5}$$

for some π'_n and $\theta_n \in [0,1)$. For convenience, further define the set $S' \subset \mathcal{S}$, which is the set of states s such that $\rho_n(s) \neq \pi(s)$.

Then neither off-policy Monte Carlo algorithm need converge:

Theorem 1. *Neither off-policy Monte Carlo algorithm following policy ρ_n need converge to the correct Q-values for π, even if the agent generates every possible episode history infinitely often.*

The proof is given in Section 3. But that convergence failure need not happen for all such ρ_n. A sufficient condition for convergence is:

Theorem 2. *Let $\sigma_n = 1/(1-\theta_n)$. Assume that σ_n is eventually non-decreasing, and that there exists a $\delta > 0$ such that, for large enough n, $(\sigma_n)^{\sqrt{\log(n)}} < n^{1-\delta}$. Assume that π visits each state-action pair with non-zero probability.*

Then both off-policy Monte Carlo algorithms following policy ρ_n will almost surely converge on the correct Q-values for π.

This will be proved in Section 4. Note that the requirement for visiting all (s,a) pairs is for the evaluation policy π – the behaviour policy will also do so, as a consequence of the conditions above. One immediate corollary of theorem 2 is:

Corollary 1. *If θ_n is bounded above by $\kappa < 1$, both off-policy Monte Carlo algorithms using ρ_n will converge on the correct Q-values for π, if π visits each state-action pair with non-zero probability.*

Proof. This kind of result has been proved before [5]. The σ_n are bounded above by $1/(1-\kappa)$, and thus, for any δ, are less than $n^{1-\delta}$ for sufficiently large n. Then Theorem 2 implies the result. □

A second corollary is:

Corollary 2. *If $\theta_n = 1 - \frac{1}{\log(n)}$, both off-policy Monte Carlo algorithms using ρ_n will converge on the correct Q-values for π, if π visits each state-action pair with non-zero probability.*

Proof. Set $\delta = 1/2$, note that $\sigma_n = \log(n)$, and that

$$\begin{aligned}\log\left((\sigma_n)^{\sqrt{\log(n)}}\right) &= \sqrt{\log(n)}\log(\log(n)) \\ &< \frac{\log(n)}{2} = \log(\sqrt{n})\end{aligned}$$

for large enough n. Hence, for large enough n, $(\sigma_n)^{\sqrt{\log(n)}} < \sqrt{n} = n^{1-1/2}$. Then Theorem 2 implies the result. □

And a third corollary is:

Corollary 3. *Assume $\pi'_n(a|s) > r_{(s,a)}$ whenever $\pi(a|s) > 0$, for constants $r_{(s,a)} > 0$. Then both off-policy Monte Carlo algorithms using ρ_n will converge on the correct Q-values for π, if π visits each state-action pair with non-zero probability.*

Proof. Let r be the minimum value of $r_{(s,a)}/\pi(a|s)$, across all (s,a) where $\pi(a|s) > 0$. Then π'_n can be rewritten as:

$$\pi'_n = r\pi + (1-r)\pi''_n,$$

for some π''_n. Similarly ρ_n can be rewritten as

$$\rho_n = (1-(1-r)\theta_n)\pi + (\theta_n)(1-r)\pi''_n.$$

Then note that $(\theta_n)(1-r)$ is bounded above by $1-r$, and the result follows by corollary 1. □

The rest of this paper will be dedicated to proving theorems 1 and 2.

3 Failure to converge

This Section aims to prove Theorem 1, by constructing a counter-example using the MDP of figure 1.

Set $\pi(a|s_1) = 1$, $\pi(a|s_2) = \pi(b|s_2) = 1/2$, $S' = \{s_2\}$, $\pi'_n(a|s_2) = 0$, $\pi'_n(b|s_2) = 1$, and define θ_n as

$$\theta_n = 1 - 1/(n\log(n)).$$

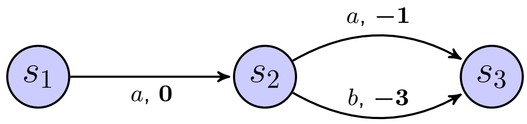

Fig. 1. An MDP where the off-policy Monte Carlo algorithms can fail to compute the correct Q-values.

The probability of the n-th episode history being $h^b = (s_1, a, 0, s_2, b, -3, s_4)$ is greater or equal to $1/2$: this history therefore almost surely gets generated infinitely often.

Now consider the episode history $h^a = (s_1, a, 0, s_2, a, -1, s_4)$. It will get generated during episode n with probability

$$\frac{1}{2} \cdot (1 - \theta_n) = \frac{1}{2n \log(n)}.$$

Each episode is independent and $\sum_n 1/(2n\log(n)) = \infty$, so by the converse Borel-Cantelli lemma, the episode is generated infinitely often.

Then note that if the history h^a is generated during the n-th episode, it is generated with weight $1/(1 - \theta_n) = n \log(n)$, while the history h^b is generated with weight $1/(1 + \theta_n) = 1/(2 - 1/(n \log(n)))$. Split the weight total $W_n(s_1, a)$ as $W_n^a(s_1, a) + W_n^b(s_1, a)$ (the weight totals due to the histories h^a and h^b respectively). Then ordinary importance sampling will give Q_n as:

$$Q_n(s_1, a) = \frac{-W_n^a(s_1, a) - 3W_n^b(s_1, a)}{N_n(s_1, a)}, \tag{6}$$

while weighted importance sampling gives:

$$Q_n(s_1, a) = \frac{-W_n^a(s_1, a) - 3W_n^b(s_1, a)}{W_n^a(s_1, a) + W_n^b(s_1, a)}. \tag{7}$$

In state s_1, both actions are equiprobable and independent, so the law of large numbers implies $N_n(s_1, a)/n \to 1$ almost surely. The $W_n^b(s_1, a)$ is the sum of weights less than 1, so $W_n^b(s_1, a) \leq N_n(s_1, a)$. The probability of h^b increases to 1, and the weights for that history are larger than $1/2$ for $n \geq 1$. Thus, almost surely, for sufficiently large n, $n/3$ and $2n$ are lower and upper bounds for both $N_n(s_1, a)$ and $W_n^b(s_1, a)$.

Now pick an n where episode history h^a is generated, which must happen infinitely often. The weight of this history is $n \log(n)$, so $W_n^a(s_1, a) \geq n$. In the limit, this must dominate the n-bounded contributions of $N_n(s_1, a)$ and $W_n^b(s_1, a)$.

Thus, for a large enough n where h^a is generated, equation (6) then gives the upper bound

$$Q_n(s_1, a) \leq -C \log(n),$$

for some constant C. Conversely, equation (7) gives an upper bound:

$$Q_n(s_1, a) \geq -1 - C/\log(n),$$

for some C.

The correct Q-value for $Q(s_1, a)$ under π is clearly $(-3-1)/2 = -2$, so neither algorithm can converge to the correct values. Ordinary importance sampling clearly cannot find the optimal policy (of choosing a in state s_2) either. If b had a reward of 0 instead of a reward of -2, then weighted importance sampling would fail to find the optimal policy on that MDP, so both algorithms can fail to find optimal policies.

Remark 1. The lack of convergence can also be proved for $\theta_n = 1 - 1/n$, but the proofs are more involved.

4 Proof of convergence

This Section aims to prove[3] Theorem 2.

4.1 Infinite variance

The reason that such mathematical machinery is needed is because, in many cases, the variance of the reward become infinite. Consider the MDP of figure 2. Two actions are available in s_2: action a which, with p probability will return the agent to state 1 with a reward of 1, and otherwise send them to state s_3 with no reward. And b, which goes straight to s_3 with no reward. Set $\pi(a|s_1) = 1$, $\pi'_n(b|s_2) = 1$, and $\theta_n = 1 - q$. The weights are powers of $1/q$, $\rho_n(a|s_2) = q$, so the expected weighted reward is the correct $\sum_{l=0}^{\infty}(l/q^l)(pq)^l$, which is finite. Since for any random variable X, $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$, the variance of the weighted reward is finite iff the expected squared weighted reward is finite.

Then the expected squared weighted reward is:

$$\sum_{l=0}^{\infty}(l/q^l)^2(pq)^l = \sum_{l=0}^{\infty} l^2(p/q)^l.$$

And this sum diverges for $q \leq p$.

4.2 Proving convergence

This proof will use ordinary importance sampling, then generalise to weighted importance sampling.

Fix any pair (s, a). Since π must visit that pair with finite probability, there is an $m < |\mathcal{S} \times \mathcal{A}|$ and a probability $\tau > 0$ such that an agent following π would

[3] The proof will closely mirror the standard proofs of the strong law of large numbers, see for instance https://terrytao.wordpress.com/2008/06/18/the-strong-law-of-large-numbers/

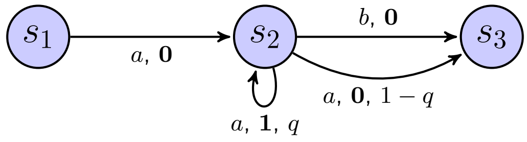

Fig. 2. An MDP where the variance of the reward can become infinite within a single episode.

reach (s, a) in m steps with probability τ. During episode n, an agent following ρ has a probability at least $\tau(1-\theta_n)^m$ of reaching (s, a). For large enough n, $\sqrt{\log(n)} > m$. Then, by the assumptions of the Theorem, for large enough n,

$$(\sigma_n)^m < (\sigma_n)^{\sqrt{\log(n)}} < n^{1-\delta} < n.$$

Hence, for large enough n, $\tau(1-\theta_n)^m = \tau(1/\sigma_n) > \tau/n$, so $\sum^{\infty} \tau(1-\theta_n)^m = \infty$. Then since each episode is independent, the converse Borel-Cantelli lemma implies that an agent following ρ will almost surely visit (s, a) infinitely often.

How regular will these visits be? The expected number of visits during the episodes up to the n-th is simply

$$\sum_{j=1}^{n} \tau(1-\theta_j)^m.$$

For large enough n, $\sigma_n < n^{\frac{1-\delta}{\sqrt{\log(n)}}}$ and so eventually

$$\tau(1-\theta_n)^m > \tau n^{-m\frac{1-\delta}{\sqrt{\log(n)}}} > n^{-\delta'},$$

for any $\delta' > 0$. Therefore, for large enough n, the number of visits to (s, a) among the n episodes must be almost surely greater than $\sum_{j=1}^{n} j^{-\delta'} = O(n^{1-\delta'})$. Thus, for large enough n, $N_n(s, a) > n^{1-\delta'}$ almost surely.

Let Q^* be the true Q-value of the MDP under π, and let $\mathcal{H}$ be the set of possible episode histories, ignoring rewards. Let $\mathcal{H}_l$ be the set of histories of length l. Write $\pi(h_n)$ for $h \in \mathcal{H}$ to designate the probability that episode n has history h if the agent follows the policy π. Note that since π is fixed, $\pi(h_n)$ is independent of n (unlike $\rho_n(h_n)$).

Let η be the policy designed to maximise the expected time the agent spends in the MDP. This means that η is a Markov policy, as the policy that maximises the time spent if the agent is in state (s', a') does not depend on the agent's prior history. Combined with the MDP, η describes a Markov chain, with absorbing final states. Its transition matrix is of the form:

$$\begin{pmatrix} P' & R \\ 0 & Id \end{pmatrix},$$

for P' a transition matrix on the non-absorbing state-action pairs, and Id the identity matrix on the absorbing ones. Since any episode must terminate with probability 1, the matrix P' has a single maximal real eigenvalue $\mu' < 1$ [2]. For large n, the probability that the matrix will not have terminated by the n-th episode is bounded by $(\mu')^n$. Since η is the policy that maximises the expected time spent in the MDP, an agent following π cannot expect to stay in MDP longer than that, so there exists a C' such that

$$\pi(\mathcal{H}_l) \leq C'(\mu')^l.$$

Fix any $\mu' < \mu < 1$, then because $l^2(\mu')^l$ must eventually be less than μ^l, there exists a C such that

$$l\pi(\mathcal{H}_l) \leq l^2\pi(\mathcal{H}_l) \leq C\mu^l.$$

Let E be the maximal expected reward the agent can generate from a single state-action pair. Let S be the maximal expected squared reward the agent can generate from a single state-action pair. Then if R_h is the random variable denoting the reward generated along history $h \in \mathcal{H}_l$,

$$\begin{aligned} \mathrm{E}\left(R_h|h\right) &\leq lE \\ \mathrm{Var}\left(R_h|h\right) \leq \mathrm{E}\left(R_h^2|h\right) &\leq l^2 S. \end{aligned}$$

Let WR_n^l denote the random variable that returns 0 if the length of the n-th episode is not l, and the (weighted) reward otherwise, *under the assumption that the agent visits* (s,a) *during episode* n. Therefore:

$$WR_n^l = \sum_{h \in \mathcal{H}_l} w(h_n) R_h \rho(h_n).$$

Note that $w(h_n) = \pi(h_n)/\rho(h_n) \leq (\sigma_n)^{i_h}$, where $i_h \leq l$ is the number of times that the agent goes through a state in S' along h. Then

$$\begin{aligned} \mathrm{E}_{\rho_n}\left(WR_n^l\right) &= \sum_{h \in \mathcal{H}_l} w(h_n) \mathrm{E}\left(R_h|h\right) \rho_n(h_n) \\ &= \sum_{h \in \mathcal{H}_l} \mathrm{E}\left(R_h|h\right) \pi(h_n), \end{aligned}$$

which is the expected reward from episode histories of length l from an agent following π. Therefore $\sum_{l=1}^{\infty} \mathrm{E}_{\rho_n}\left(WR_n^l\right) = \sum_h \mathrm{E}\left(R_h|h\right)\pi(h_n) = Q^*$.

The expectation and variance of WR_n^l can be bounded as:

$$\begin{aligned}
\mathrm{E}_{\rho_n}\left(WR_n^l\right) &= \sum_{h\in\mathcal{H}_l} \mathrm{E}\left(R_h|h\right)\pi(h_n) \\
&\leq lE\pi(\mathcal{H}_l) \\
&\leq EC\mu^l \\
\mathrm{Var}_{\rho_n}\left(WR_n^l\right) &\leq \mathrm{E}_{\rho_n}\left(\left(WR_n^l\right)^2\right) \\
&\leq \sum_{h\in\mathcal{H}_l} \mathrm{E}\left(\left(w(h_n)R_h\right)^2|h\right)\rho(h_n) \\
&\leq \sum_{h\in\mathcal{H}_l} (\sigma_n)^{i_h}\mathrm{E}\left(\left(R_h\right)^2|h\right)\pi(h_n) \\
&\leq \pi(\mathcal{H}_l)(\sigma_n)^l l^2 S \\
&\leq SC(\mu\sigma_n)^l.
\end{aligned} \tag{8}$$

Redefine C as $\max(EC, SC)$ so that these bounds are $C\mu^l$ and $C(\mu\sigma_n)^l$ respectively.

Then define $WR_n^{<l} = \sum_{j<l} WR_n^j$ and $WR_n^{\geq l} = \sum_{j\geq l} WR_n^j$. Equation (8) then implies that

$$\begin{aligned}
\mathrm{E}_{\rho_n}\left(WR_n^{\geq l}\right) &\leq C\sum_{j=l}^{\infty}\mu^j \\
&\leq \frac{C\mu^l}{1-\mu} \\
\mathrm{Var}_{\rho_n}\left(WR_n^{<l}\right) &\leq \sum_{j=1}^{l-1}\mathrm{Var}_{\rho_n}\left(WR_n^{<l}\right) \\
&\leq \sum_{j=1}^{l-1} C(\mu\sigma_n)^j \\
&\leq \frac{C(\sigma_n\mu)^l - 1}{\sigma_n\mu - 1} \\
&\leq \frac{C(\sigma_n)^l}{\sigma_n\mu - 1}
\end{aligned}$$

Redefine C as $C/(1-\mu)$ so that the first bound is $C\mu^l$. For large enough n, $\sigma_n\mu - 1 > 1$; so, redefining C if needed to cover the finitely many smaller values of n, the second bound is $C(\sigma_n)^l$:

$$\begin{aligned}
\mathrm{E}_{\rho_n}\left(WR_n^{\geq l}\right) &\leq C\mu^l \\
\mathrm{Var}_{\rho_n}\left(WR_n^{<l}\right) &\leq C(\sigma_n)^l
\end{aligned}$$

Let I_n be the indexing variable that denotes that the agent visited (s,a) during episode n. By assumption, $N_n(s,a) = \sum_{j=1}^{n} I_n(s,a)$, and this section has shown that $N_n(s,a) > n^{1-\delta'}$ for any $\delta' > 0$ and large enough n. Define

$$Q_n^{\geq l} = \frac{1}{N_n(s,a)} \sum_{j=1}^{n} I_j W R_j^{\geq l}$$
$$Q_n^{< l} = \frac{1}{N_n(s,a)} \sum_{j=1}^{n} I_j W R_j^{< l}.$$

Then, since σ_n are eventually non-decreasing, for any $\delta' > 0$ and for large enough n:

$$\mathrm{E}_{\rho_n}\left(Q_n^{\geq l}\right) \leq \frac{1}{N_n(s,a)} \sum_{j=1}^{n} I_j C \mu^l \leq \frac{\sum_{j=1}^{n} I_j}{N_n(s,a)} C\mu^l \leq C\mu^l$$
$$\mathrm{Var}_{\rho_n}\left(Q_n^{< l}\right) \leq \frac{1}{N_n(s,a)^2} \sum_{j=1}^{n} C I_j (\sigma_j)^l \leq \frac{C}{N_n(s,a)} (\sigma_n)^l \leq \frac{C}{n^{1-\delta'}} (\sigma_n)^l.$$

The following is a key Lemma:

Lemma 1. *Let $1 \leq m_1 \leq m_2 \ldots$ be a sequence that is lacunary in that there exists a $c > 1$ such that $m_{j+1}/m_j > c$ for sufficiently large j. Then Q_{m_j} converges to Q^* almost surely as $j \to \infty$.*

Proof. Fix any $\epsilon > 0$, and consider the subsequence $n_j = \lceil e^{\sqrt{j}} \rceil$. Then since $\{m_j\}$ is eventually an exponentially growing sequence, $m_j > n_j$ for sufficiently large j.

Set $l_j = \lfloor \sqrt[4]{j} \rfloor$ and consider $Q_{m_j} = Q_{m_j}^{\geq l_j} + Q_{m_j}^{< l_j}$. Then note that if $k = \sqrt[4]{j}$, $\mathrm{E}\left(Q_{m_j}^{\geq l_j}\right) \leq C\mu^{l_j}$, and

$$\frac{j^2}{\epsilon^2} C\mu^{l_j} \leq \frac{k^8}{\epsilon^2} C\mu^{k-1} \to 0,$$

as $k \to \infty$. This implies that for large enough j, $\mathrm{E}\left(Q_{m_j}^{\geq l_j}\right) \leq \frac{\epsilon}{j^2}$.

Any random variable X with finite expectation has the first moment bound:

$$P(|X| \geq \lambda) \leq \frac{\mathrm{E}|X|}{\lambda}.$$

Setting $\lambda = \epsilon$, this implies that $P(|Q_{m_j}^{\geq l_j}| \geq \epsilon) \leq \frac{1}{j^2}$ for large enough j. Since $\sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$, the Borel-Cantelli lemma implies that, almost surely, $|Q_{m_j}^{\geq l_j}| \geq \epsilon$ only finitely often. Since $Q^* = \mathrm{E}\left(Q_{m_j}\right) = \mathrm{E}\left(Q_{m_j}^{\geq l_j}\right) + \mathrm{E}\left(Q_{m_j}^{< l_j}\right)$, this also implies $|\mathrm{E}\left(Q_{m_j}^{< l_j}\right) - Q^*| \geq \epsilon$ only finitely often.

Consider now $\mathrm{Var}\left(Q_{m_j}^{< l_j}\right) \leq \frac{C}{(m_j)^{1-\delta'}} (\sigma_{m_j})^{l_j}$.

Note that $l_j = \lfloor \sqrt[4]{j} \rfloor \leq \sqrt{\sqrt{j}} \leq \sqrt{\log(n_j)} \leq \sqrt{\log(m_j)}$. So, by the assumption of Theorem 2, for large enough j, $(\sigma_{m_j})^{l_j} < m_j^{1-\delta}$ for some $\delta > 0$.

Fix $\delta' = \delta/2$. Hence for large enough j, $\frac{C}{(m_j)^{1-\delta/2}}(\sigma_{m_j})^{l_j} < \frac{1}{(m_j)^{\delta/2}}$. Since $m_j^{\delta/2} > n_j^{\delta/2} \geq e^{(\delta/2)\sqrt{j}} > Cj^2/\epsilon^2$ for large enough j, eventually

$$\mathrm{Var}\left(Q_{m_j}^{<l_j}\right) \leq \frac{\epsilon^2}{j^2}.$$

Any random variable X with finite variance has the second moment bound:

$$P(|X - \mathrm{E}(X)| \geq \lambda) \leq \frac{\mathrm{Var}(X)}{\lambda^2}.$$

Setting $\lambda = \epsilon$, this implies that for large enough j,

$$P(|Q_{m_j}^{<l_j} - \mathrm{E}\left(Q_{m_j}^{<l_j}\right)| \geq \epsilon) \leq \frac{1}{j^2}.$$

Using the Borel-Cantelli lemma a second time, $|Q_{m_j}^{<l_j} - \mathrm{E}\left(Q_{m_j}^{<l_j}\right)| < \epsilon$ except finitely often.

Putting together the three ϵ bounds completes the proof by showing that, almost surely, except for finitely many j,

$$|Q_{m_j} - Q^*| < 3\epsilon.$$

Since ϵ was arbitrary, this completes the proof. $\square$

The next step is to generalise from lacunary sequences to all sequences. Since all rewards have been assumed positive, $nQ_n \leq mQ_m$ for $n < m$. Then for any $1 > \epsilon > 0$, define the lacunary sequence $m_j = \lceil (1+\epsilon)^j \rceil$. For $m_j < n < m_{j+1}$,

$$\lceil (1+\epsilon)^j \rceil Q_{m_j} \leq nQ_n \leq Q_{m_{j+1}} \lceil (1+\epsilon)^{j+1} \rceil Q_{m_{j+1}}.$$

This implies that for large enough j, $Q_n \geq Q_{m_j}(1/(1+\epsilon) - \epsilon^2) \geq Q_{m_j}(1-\epsilon)$ and $Q_n \leq Q_{m_j}((1+\epsilon)+\epsilon^2) \leq Q_{m_j}(1+2\epsilon)$, where the ϵ^2 term comes from the fact that $(1+\epsilon)^j$ need not be an integer. By the Lemma above, the sequence Q_{m_j} converge almost surely to Q^*, thus, for large enough j and $n > \lceil (1+\epsilon)^j \rceil$, Q_n must be within $3\epsilon(Q^*+1)$ of Q^*. Since ϵ was arbitrary, this proves that Q_n converges to Q^* almost surely as $n \to \infty$.

This completes the proof for all MDP's with positive rewards. Note that the same proof works for MDP's with negative rewards. Then the general proof is established for by dividing the rewards into positive and negative parts, noting their separate convergence, and noting that the Q-values update process is linear in rewards.

Since this proves the convergence of $Q_n(s,a)$ to $Q^*(s,a)$ for any (s,a), and there are finitely many (s,a) pairs, this proves the almost sure convergence of Q_n to Q^* in general.

It is now necessary to extend the result to weighted importance sampling. To do that, it will suffice to show that, under the conditions above, $W_n(s,a)/N_n(s,a) \to 1$ almost surely.

To see this, change the MDP by setting all rewards to 0, except for the final reward when the agent reaches a terminal state, where they will get 1. This means that the reward along each history is 1, and the weighted reward is just the weight. Thus the new Q'_n for ordinary importance sampling is $Q'_n(s,a) = W_n(s,a)/N_n(s,a)$. By the result we've just proved, these must converge almost surely to the correct Q-values for the modified MDP, *i.e.*, to 1. This proves that the ratios converge almost surely to 1, as required.

A Note on the independence assumption

The policy ρ_n, as defined in equation (5), assumes that the agent choose independently between π and π'_n for each $s \in \mathcal{S}$. If this independence is dropped, the situation can get even worse for convergence – the agent may fail to converge to the right values[4] even for fixed $\theta < 1$.

Consider the MDP in figure 3. Define the policy π as choosing randomly amongst the two actions at s_1, and choosing a otherwise. Define $S' = \{s_2, s_3\}$ and π'_n as choosing b from all states in S'.

For ρ, the probabilities of choosing π and π'_n in S are equal (and the same from episode to episode), $\theta_2 = \theta_3 = 0.5$, but they are strictly anti-correlated within a given episode.

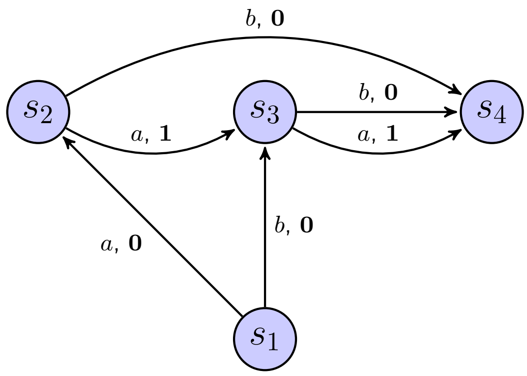

Fig. 3. An MDP on which the agent has non-Markov policy choices.

Notice that the episode history $(s_1, a, 0, s_2, a, 1, s_3, a, 1, s_4)$ *never appears.* This is because the uses of π'_n at s_2 and s_3 are anti-correlated, so the agent cannot avoid both of them. Therefore the agent never experiences a total reward of 2; moreover, the only episodes with rewards are the (equiprobable)

[4] For non-Markov policies like this one, the off-policy algorithm has to be adjusted to consider ratios $\pi(h)/\rho(h)$ for entire histories h (rather the product of state-action pair probabilities), but this is not a large change.

$(s_1, a, 0, s_2, a, 1, s_3, b, 0, s_4)$ and $(s_1, b, s_3, a, 1, s_4)$, both with reward 1. Therefore the agent will compute $Q(s_2, a)$ and $Q(s_3, a)$ as having the same values. However, it is clear that if following π, $Q(s_2, a) = 3/2$ and $Q(s_3, a) = 1$.

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